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DAMAGE SPREADING IN THE KAWASAKI ISING MODEL

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We observe no phase transition in the damage spreading probability at the Curie point of the 2D Ising model with nearest neighbor spin exchange dynamics. Furthermore, we report values of the growth exponent z close to unity, implying "speed of light" transmission of information, and values of the exponents describing the scaling behavior of the damage mass with time and distance, close to two. Finally, we discuss the spreading of damage in a one-dimensional system.

1. Introduction

Damage spreading is the study of how two initially identical systems evolve in time while subject to identical dynamics after one of them suffers a small, localized perturbation [1]. It is known that for ferromagnetic Ising models, the question of whether or not the two systems will eventually become identical again (no damage), or remain a fixed distance apart in phase space (finite, localized damage), or become largely uncorrelated (damage spreading through the entire system) depends intimately upon both the temperature of the system and the type of dynamics used. Specifically, for dynamics that involve the flipping of single spins via the Monte Carlo heat bath (probability of spin up) or Glauber (probability of spin flip) methods there exists a phase transition in the probability that the damage spreads throughout the system as a function of temperature. In fact, for heat bath dynamics, it has been shown [2, 3] that the damage in the system can be related to equilibrium thermodynamic quantities, and also tells us something about the way in which information propagates through the system.

The Kawasaki method is a type of Monte Carlo dynamics that involves the *exchange* of (in this case) nearest-neighbor spins of opposite orientation [4].

One can therefore think in terms of the diffusion of single spins through the system. Because clusters can diffuse only through the movement of their boundaries, equilibration of the system takes orders of magnitude longer than equilibration via other single-spin flip dynamics like heat bath or Glauber dynamics. Consequently, one might expect that the propagation of damage through an Ising system via Kawasaki dynamics might mimic the slow diffusion exhibited by the spins themselves. In this paper we show that this is, in fact, not the case.

2. Method

Using a Connection Machine 2 and a Sun workstation we simulated 2D square lattices of sizes ranging from 16×16 to 512×512. The lattices were equilibrated via either Swendsen–Wang dynamics [5] or long-range Kawasaki exchange [6], both of which are significantly faster than the traditional nearest-neighbor Kawasaki dynamics. Then a copy of the equilibrated lattice is made, and the central spin is flipped in the replica lattice[#1]. After this initial damage is introduced, the two lattices are updated simultaneously with nearest-neighbor Kawasaki dynamics, using the same random numbers. The use of identical random numbers is crucial to ensure that any difference between the two systems is due solely to the initial perturbation and not to different interactions with the heat bath.

After each Monte Carlo step, a logical XOR operation between the Boolean representations of lattice A and lattice B gives us the damage in the system. To measure both the damage and the touching time as a function of distance from the center, we construct "fences" every 25 lattice spacings, and when the damage touches any one of the four sides of the fence, we mark the time and the total damage in the system. These results are then averaged over many trials at the same temperature and magnetization.

An alternative procedure is to consider different lattice sizes and measure the corresponding quantities when the damage reaches the edge of the system. The results that we present here come from a combination of the two approaches.

[#1] Because Kawasaki dynamics conserves the order parameter M, flipping the central spin in fact changes M in the replica lattice from that in the original lattice by a very small amount ($2/512 \times 512$, for example, for larger lattices). A more "proper" method would involve the *exchange* of the central spin with one of its nearest neighbors as the initial perturbation. After disregarding runs in which the initial damage "healed", we found no difference in any of the results. Hence for purposes of computational efficiency, we use the original method of flipping the central spin for the remainder of our study.

3. Results

3.1. Nonexistence of phase transition

We find that, unlike damage spreading in Ising models with heat bath or Glauber dynamics, there is no phase transition with Kawasaki dynamics. That is, the damage spreads to the edges of the lattice for all lattice sizes at all temperatures, and specifically with no anomalous behavior at the Curie point. This is similar to the behavior observed for damage spreading with Swendsen–Wang dynamics [7].

3.2. How τ scales with system size

We measure the touching time, τ, for the damage to touch one of the system edges, and find

$$\langle \tau \rangle \sim L^z, \qquad z = 1.00 \pm 0.07, \tag{1}$$

for all temperatures. The data for both methods is displayed in fig. 1.

3.3. How damage mass scales with touching time

We measure the average amount of actual damage $\langle s \rangle$ in the system when

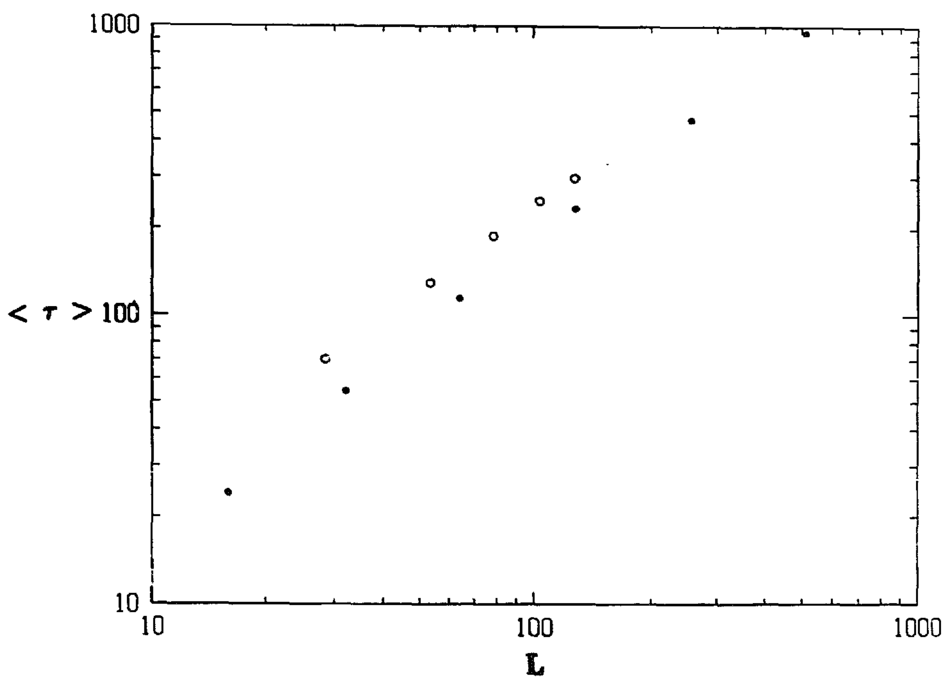

Fig. 1. Plot of average touching times versus lattice size (filled circles) and average touching times versus distance from the origin for a 256 × 256 lattice (empty circles).

the outermost damaged site touched each successive fence, as described in section 2, and also when it touched the system edge, as a function of system size, as prescribed by the alternative procedure. We find the following scaling relation:

$$\langle s\rangle \sim \tau^{\alpha}\,, \qquad \alpha = 2.3 \pm 0.3\,. \tag{2}$$

However, we expect a value of alpha bounded above by two, since the amount of damage in a 2D system cannot increase any faster than the square of the time, if the distance is increasing linearly with the time. Indeed, as we consider time intervals successively closer to the touching time (which corresponds to using only those fences closer to the edges of the lattice) we do find that α approaches 2, as can be seen in the inset of fig. 2. What this implies is very subtle, but important: The outer edges of the "damage cloud", which grow outward from the central site, reach the boundary before the interior of the cloud "fills in"; i.e. the damage growth is not homogeneous. This phenomenon is further illustrated by the exponent d_{act}, using the notation of Stauffer [1], Stanley et al. [1] and Poole and Jan [3], which describes how the damage mass scales with system size.

It is straightforward to show that only two of the three exponents described

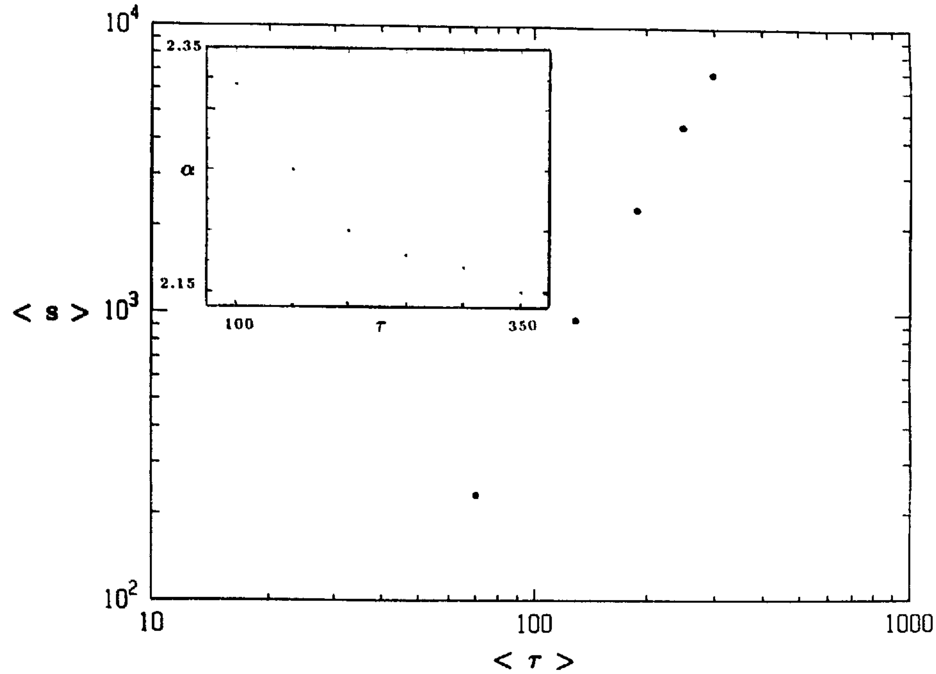

Fig. 2. Plot of average number of damaged sites versus touching time. Inset: Plot of the growth exponent α versus time as the times used get larger. When times are long enough that the growth becomes homogeneous, α approaches 2.

here are independent. By combining eqs. (1) and (2), we find the scaling relation

$$\langle s\rangle \sim L^{d_{\rm act}}, \qquad d_{\rm act} = z\alpha = 2.3 \pm 0.3, \tag{3}$$

from our values of z and α quoted above. Direct measurements of $d_{\rm act}$ by measuring the average amount of damage $\langle s\rangle$ in the system when the outermost damaged site is a fence distance L' from the center yield

$$d_{\rm act} = 2.27 \pm 0.23$$

consistent with the calculated value. This implies that the fractal dimension of the damage cloud is *greater* than the embedding space. This is misleading, however, in that the damage at the outer edges of the damage cloud travels to the boundaries at the speed of light in our system, while the inner core of the cloud "fills in" at a slower rate. Hence, upon measuring a growing object whose different parts are expanding at different rates, one obtains an exponent that does not correspond to the fractal dimension in the normal sense of the word. Indeed, if we wait until the damage has reached an equilibrium value, and measure the equilibrated damage as a function of system size, we find the expected value of $d_{\rm act} = 2$.

3.4. Damage in one dimension

To help us understand the 2D results, let us consider a 1D chain of Ising spins with Kawasaki dynamics. The critical temperature for this system is at $T = 0$, and for a system with zero magnetization the ground state is a semi-infinite domain of up spins and a semi-infinite domain of down spins. Imagine now that we perturb the system by flipping one of the spins at the edge of its domain – we say that the damage is created at the boundary of the domains. Because the system is at zero temperature, nothing will happen. However, if the damage is created within a domain then the damage will diffuse as a random walk until it reaches the edge of the domain. We find that $z = 2$ for this period of diffusion.

At low temperatures $T > 0$ we find that $z = 1$ and $d_{\rm act} \simeq 1$. We can understand this result by considering a temperature such that the system has domains with average size l. Then, on average, we require l attempts for the damage initiated at a domain wall to propagate a distance of l, and consequently $z = 1$ as we confirm with Monte Carlo simulations. Our numerical results also give a value of $d_{\rm act} \simeq 1.12$, but we expect this to approach 1 as the system size tends to larger values. This is consistent with our findings in two dimensions.

3.5. Damage in equilibrium

After some time, the damage saturates at a value $\langle s \rangle_\infty$. We find that, although α appears to be temperature independent, the crossover time from power-law growth to saturation *does* depend somewhat on the temperature, taking much longer to saturate at low temperatures than at high temperatures. Moreover, the amount of damage per site at saturation can be related to the magnetization per site of the system at a given temperature.

Consider the two lattices after the damage has spread completely through the system. The two lattices are now uncorrelated and independent. The probability that site i is damaged is simply the probability that site i is up in lattice A and down in lattice B, or vice versa:

$$\langle s \rangle_\infty = p_\uparrow^A p_\downarrow^B + p_\downarrow^A p_\uparrow^B = \frac{N_\uparrow}{N}\frac{N_\downarrow}{N} + \frac{N_\downarrow}{N}\frac{N_\uparrow}{N}, \tag{4}$$

where $N_\uparrow$ is the number of up spins in one lattice, $N_\downarrow$ is the number of down spins, and N is the total number of spins.

Now, the magnetization per site is simply the difference between the number of up spins and the number of down spins, divided by the total number of spins:

$$m = \frac{N_\uparrow - N_\downarrow}{N} = \frac{2N_\uparrow}{N} - 1 = 1 - \frac{2N_\downarrow}{N}. \tag{5}$$

By rearranging eq. (5) and substituting into eq. (4), we obtain the result

$$\langle s \rangle_\infty = \tfrac{1}{2}(1 - m^2), \tag{6}$$

which relates the saturation damage to the magnetization of the system. We note that this result agrees numerically with those found by Stauffer for damage spreading with Swendsen–Wang dynamics [7]. As one might expect, $\langle s \rangle_\infty = \frac{1}{2}$ when $m = 0$, and approaches zero for $m = 1$ in the limit of large N.

4. Conclusion

We have found for the spin-exchange Ising model that damage always spreads through the system, regardless of the temperature. Moreover, we have observed speed of light transmission of the boundary of the damage cloud at all $T > 0$ in both one and two dimensions. The "core" of the cloud filled in more slowly than the boundary of the cloud grew to the edges, and this gave values of the growth exponent α and the fractal dimension d_{act} greater than two, which approached 2 after adjustments were made to account for the nonhomogeneous growth of the damage cloud.

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